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Physical Theory of the Electric Wave-Filter

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NOTE: The electric wave-filter, an invention of Dr. Campbell, is one of the most important of present day circuit developments, being indispensable in many branches of electrical communication. It makes possible the separation of a broad band of frequencies into narrow bands in any desired manner, and as will be gathered from the present article, it effects the separation much more sharply than do tuned circuits. As the communication art develops, the need will arise to transmit a growing number of telephone and telegraph messages on a given pair of line wires and a growing number of radio messages through the ether, and the filter will prove increasingly useful in coping with this situation. The filter stands beside the vacuum tube as one of the two devices making carrier telegraphy and telephony practicable, being used in standard carrier equipment to separate the various carrier frequencies. It is a part of every telephone repeater set, cutting out and preventing the amplification of extreme line frequencies for which the line is not accurately balanced by its balancing network. It is being applied to certain types of composited lines for the separation of the d.c. Morse channels from the telephone channel. It is finding many applications to radio of which multiplex radio is an illustration. The filter is also being put to numerous uses in the research laboratory.

The present paper is the first of a series on the electric wave-filter to be contributed to the Technical Journal by various authors. Being an introductory paper the author has chosen to discuss his subject from a physical rather than mathematical point of view, the fundamental characteristics of filters being deduced by purely physical reasoning and the derivation of formulas being left to a mathematical appendix.—*Editor.*

THE purpose of this paper is to present an elementary, physical explanation of the wave-filter as a device for separating sinusoidal electrical currents of different frequencies. The discussion will be general, and will not involve assumptions as to the detailed construction of the wave-filter; but in order to secure a certain numerical concreteness, curves for some simple wave-filters will be included. The formulas employed in calculating these curves are special cases of the general formulas for the wave-filters which are, in conclusion, deduced by the method employed in the physical theory.

All the physical facts which are to be presented in this paper, together with many others, are implicitly contained in the compact formulas of the appendix. Although only comparatively few words of explanation are required to derive these formulas, they will not be presented at the start, since the path of least resistance is to rely implicitly upon formulas for results, and ignore the troublesome question as to the physical explanation of the wave-filter. In order to examine directly the nature of the wave-filter in itself, as a physical structure, we proceed as though these formulas did not exist.

It is intended that the present paper shall serve as an introduction to important papers by others in which such subjects as transients on wave-filters, specialized types of wave-filters, and the practical design of the most efficient types of wave-filters will be discussed.[1]

Definition of Wave-Filter

A wave-filter is a device for separating waves characterized by a difference in frequency. Thus, the wave-filter differentiates between certain states of motion and not between certain kinds of matter, as does the ordinary filter. One form of wave-filter which is well known is the color screen which passes only certain bands of light frequencies; diffraction gratings and Lippmann color photographs also filter light. Wave-filters might be constructed and employed for separating air waves, water waves, or waves in solids. This paper will consider only the filtering of electric waves; the same principles apply in every case, however.

In its usual form the electric wave-filter transmits currents of all frequencies lying within one or more specified ranges, and excludes currents of all other frequencies, but does not absorb the energy of these excluded frequencies. Hence, a combination of two or more wave-filters may be employed where it is desired to separate a broad band of frequencies, so that each of several receiving devices is sup-

[1] I take pleasure in acknowledging my indebtedness to Mr. O. J. Zobel for specific suggestions, and for the light thrown on the whole subject of wave-filters by his introduction of substitutions which change the propagation constant without changing the iterative impedance.

plied with its assigned narrower range of frequencies. Thus, for instance, with three wave-filters the band of frequencies necessary for ordinary telephony might be transmitted to one receiving device, all lower frequencies transmitted to a second device, and all higher frequencies transmitted to a third device—separation being made without serious loss of energy in any one of the three bands.

By means of wave-filters interference between different circuits or channels of communication in telephony and telegraphy, both wire and radio, can be reduced provided they operate at different frequencies. The method is furthermore applicable, at least theoretically, to the reduction of interference between power and communication circuits. The same is true of the simultaneous use of the ether, the earth return, and of expensive pieces of apparatus employed for several power or communication purposes. In all cases the principle involved is the same as that of confining the transmission in each circuit or channel to those frequencies which serve a useful purpose therein and excluding or suppressing the transmission of all other frequencies. In the future, as the utility of electrical applications becomes more widely and completely appreciated, there will be an imperative necessity for more and more completely superposing the varied applications of electricity; it will then be necessary, to avoid interference, to make the utmost use of every method of separating frequencies including balancing, tuning, and the use of wave-filters.

Definition of Artificial Line

The wave-filter problem in this paper is discussed as a phase of the artificial line problem, and it is desirable to start with a somewhat generalized definition of the artificial line. The definition will, however, not include all wave-filters or all artificial lines, since a perfectly general definition is not called for here. Even if an artificial line is to be, under certain wave conditions, an imitation of, or a substitute for, an actual line connecting distant points, hardly any limitation is thereby imposed upon the structure of the device; an actual line need not be uniform but may vary abruptly or gradually along its length and may include two, three, four or more transmission conductors of which one may be the earth. Having indicated that wave-filters partake of somewhat this same generality of structure, the present paper is restricted to wave-filters coming under the somewhat generalized artificial line specified by the following definition:

An artificial line is a chain of networks connected together in sequence through two pairs of terminals, the networks being identical but other-

wise unrestricted. This generalized artificial line possesses the well-known sectional artificial line structure but it need not be an imitation of, or a substitute for, any known, real, transmission line connecting together distant points. The general artificial line is shown by Fig. 1 where N, N,....are the identical unrestricted networks which may contain resistance, self-inductance, mutual inductance, and capacity.

In discussing this type of structure as a wave-filter, the point of view of an artificial line is adopted for the reason that it is advantageous to regard the distribution of alternating currents as being dependent upon both propagation and terminal conditions, which are to be separately considered. In this way the attenuation, or

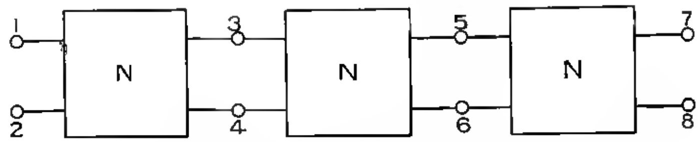


Fig. 1—Generalized Artificial Line as Considered in the Present Paper, where N, N, . . . are Identical Arbitrary Electrical Networks

falling off, of the current from section to section may be most directly studied. Terminal effects are not to be ignored, but are allowed for, after the desired attenuation effects have been secured, possibly by an increase in the number of sections to be employed.

The fundamental property of this generalized artificial line, which includes uniform lines as a special case, is the mode in which the wave motion changes from one section to the next, and may be stated as follows:

Wave Propagation Theorem

Upon an infinite artificial line a steady forced sinusoidal disturbance falls off exponentially from one section to the next, while the phase changes by a constant amount. Reversing the direction of propagation does not alter either the attenuation or phase change. When complex quantities are employed the exponential includes the phase change.[2] This theorem is proved, without mathematical equations, by observing

[2] This theorem is not new, but it is ordinarily derived by means of differential or difference equations whereas it may be derived from the most elementary general considerations, thus avoiding all necessity of using differential or difference equations, as illustrated in my paper "On Loaded Lines in Telephonic Transmission" (*Phil. Mag.*, vol. 5, pp. 313–331, 1903). In that discussion, as well as in this present one, it is tacitly assumed that the line is either an actual line with resistance, or the limit of such a line as the resistance vanishes, so that the amplitude of the wave never increases towards the far end of an infinite line.

that the percentage reduction in amplitude and the change in phase, in passing from the end of one section to the corresponding point of the next section, do not depend upon either the absolute amplitude or phase; they depend, instead, only upon the magnitudes, angles and interconnections of the impedances between the two points and of the impedances beyond the second point. These impedances are, since the line is assumed to be periodic and infinite, identically the same for corresponding points between all sections of the line, and, therefore, the relative changes in the wave will be identical at corresponding points in all sections. This proves the exponential falling off of the disturbance and the constancy of phase change; the ordinary reciprocal property shows that the wave will fall off identically whichever be the direction of propagation. By the superposition property it follows that the steady state on any finite portion of a periodic recurrent structure must be the sum of two equally attenuated disturbances, one propagated in each direction.

The fundamental wave propagation theorem may be generalized for any periodic recurrent structure irrespective of the number and kind of connections between periodic sections, provided the disturbance is such as to remain similar to itself at corresponding points of each of these connections.

Equivalent Generalized Artificial Line

Since, at a given frequency, any network employed solely to connect a pair of input terminals with a pair of output terminals may be replaced by either three star-connected impedances or three delta-connected impedances, the general artificial line of Fig. 1 may be

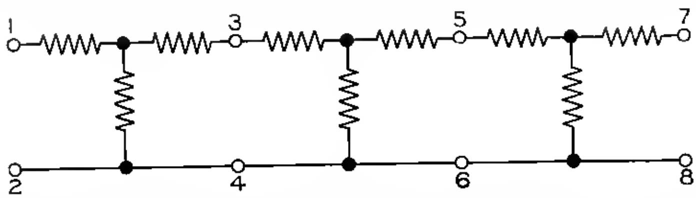


Fig. 2—Equivalent Artificial Line Obtained by Substituting Star Impedances

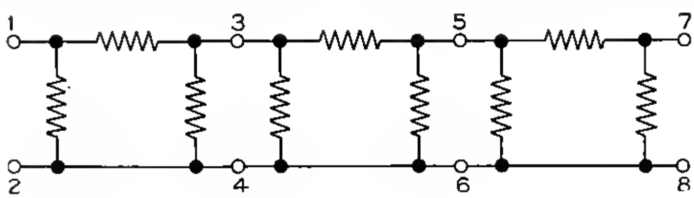


Fig. 3—Equivalent Artificial Line Obtained by Substituting Delta Impedances

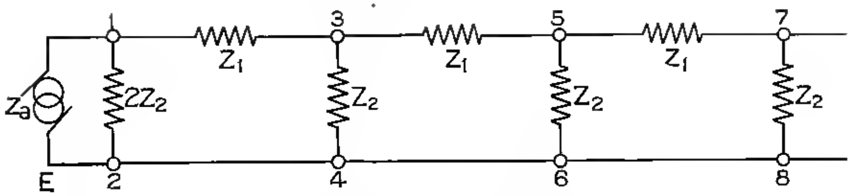


Fig. 4—Equivalent Ladder Artificial Line

replaced by the equivalent artificial line of either Fig. 2 or Fig. 3. By combining the series impedances in Fig. 2 and the parallel impedances in Fig. 3, the equivalent line in Fig. 4 is obtainable. The two ways of arriving at Fig. 4 give different values for the series and shunt impedances Z_1, Z_2, and different terminations for the line, but the propagation of the wave is the same in both cases, since the assumed substitutions are rigorously exact. While Fig. 4 may be considered as the generalized artificial line equivalent to Fig. 1, this requires including in Z_1 and Z_2 impedances which cannot always be physically realized by means of two entirely independent networks, one of which gives Z_1 and the other Z_2. This restriction is of no importance when we are discussing the behavior of the generalized artificial line at a single frequency; accordingly, the ladder artificial line is suitable for this part of the discussion. When we come to the more specific correlation of the behavior of the generalized artificial line at different frequencies, it will be found more convenient to replace the ladder artificial line by the lattice artificial line, which avoids the necessity of considering any impedances which are not individually physically realizable.

The equivalence between Figs. 1 and 4 is implicitly based upon the assumption that it is immaterial, for artificial line uses, what absolute potentials the terminals 1, 2; 3, 4; 5, 6; etc. have—this leaves us at liberty to connect 2, 4, 6, etc., together, so long as we maintain unchanged the differences in potential between 1 and 2, 3 and 4, etc. Instead of connecting 2 and 4 we might equally well connect 2 and 3, and then Z_1 would connect 1 and 4 as in Fig. 5; with these

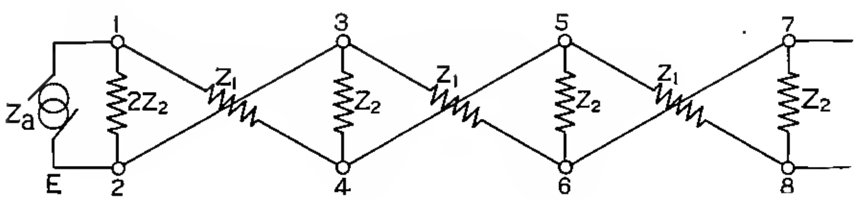


Fig. 5—Equivalent Artificial Line with Crossed Impedances

cross-connections the propagation still remains unchanged. We have again obtained Fig. 4 with no circuit difference except the interchange of terminals 3 and 7 with terminals 4 and 8; or, if this is ignored, a reversal in the sign of the current at alternate pairs of terminals. This shows that the reversal of the current in alternate sections of Fig. 4 may not be of primary significance, since networks which are essentially equivalent have reversed currents.

In order to deal, at the start, with only the simpler terminal conditions, we may consider the line to begin with only one-half of the series impedance Z_1, or only one-half of the bridged admittance $1/Z_2$. These mid-points are called the mid-series and mid-shunt points; knowing the results of termination at either of these points, the effect of termination at any other point may be readily determined. For Fig. 4 termination at mid-shunt has been chosen so that each section of the line adds a complete symmetrical mesh to the network.

An alternator, introducing an impedance Z_a, is shown as the source of the steady-state sinusoidal current in Fig. 4. Assume that the impedance Z_a is variable at pleasure, and that it is gradually adjusted to make the total impedance in the generator circuit vanish,—in this case no e.m.f. will be required to maintain the forced steady-state which becomes a free oscillation. If, in addition, it is assumed that the line has an infinite number of sections, this required value of Z_a will be the negative of the mid-shunt iterative impedance[3] of the artificial line, which will be designated as K_2. The first shunt on the line now includes $-K_2$ in parallel with $2Z_2$ so that its total impedance is, say, $Z' = -2Z_2K_2/(2Z_2-K_2)$. The infinite line with its first shunt given the special value Z' is thus capable of free oscillation.

It is possible to simplify this infinite oscillating circuit by cutting off any part of it which has the same free period as the whole circuit. The entire infinite line beyond the second shunt 3, 4 certainly has this same free period, provided its first shunt also has the impedance Z'. Conceive the shunt Z_2 at 3, 4 as replaced by the four impedances $2Z_2$, $2Z_2$, $+K_2$ and $-K_2$ all in parallel; the first and last, which together make the Z' required by the infinite line, leave $2Z_2$ and

[3] The "iterative impedance" of an artificial line is the impedance which repeats itself when one or more sections of the artificial line are inserted between this impedance and the point of measurement. It is thus the impedance of an infinite length of any actual artificial line, regardless of the termination of the remote end of the line. In general, its value is different for the two directions of propagation, but not when the line is symmetrical, as at mid-series and mid-shunt. The values at these points are denoted by K_1 and K_2. "Iterative impedance" is employed because it is a convenient term which is distinctive and describes the most essential property of this impedance; it seems to be more appropriate than "characteristic impedance," "surge impedance" and the other synonyms in use.

$+K_2$ in parallel, which have the impedance $Z''=+2Z_2K_2/(2Z_2+K_2)$. Removing Z' together with the infinite line on the right there remains on the left a closed circuit made up of the three impedances Z_1, Z' and Z'' in series.

After the division, the infinite line on the right will continue, without modification, to oscillate freely, since it is an exact duplicate of the original oscillating line, and so must maintain the free oscillation already started. Since it oscillates freely by itself, it had originally no reaction upon the simple circuit from which it was separated; this simple circuit on the left must thus also continue its own free oscillations without change in period or phase.

We might continue and subdivide the entire infinite line into identical simple circuits but it is sufficient to consider this one detached circuit, which is shown separately in two ways by Fig. 6, since from

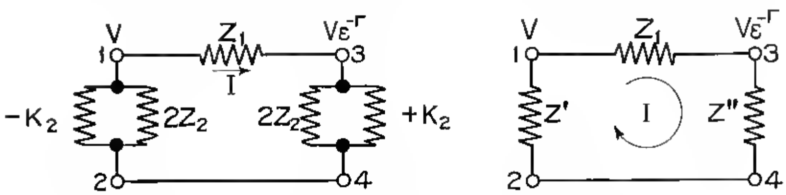


Fig. 6—Equivalent Section of Fig. 4 Terminated for Free Oscillation

its free oscillations the mathematical formulas for the steady-state propagation in the artificial line may be derived. This is deferred, however, until after the physical discussion is completed, so as to leave no room for doubt that the essentials of the physical theory are really deduced without the aid of mathematical formulas.

The generalized artificial line, if made up entirely of pure resistances, will attenuate all frequencies alike, and the entire wave will be in the same phase; this remains true, whatever be the impedance of the individual branch of the network, provided the ratio of the impedances of all branches is a constant independent of the frequency. This is precisely the condition to be avoided in a wave-filter; branches must not be similar but dissimilar as regards the variation of impedance with frequency. This calls for inductance and capacity with negligible resistance, so that there is an opportunity for the positive reactance of one branch to react upon the negative reactance of another branch, in different proportions at different frequencies. Assuming the unit network N of Fig. 1 to be made up of a finite number of pure reactances, the equivalent impedances Z_1 and Z_2 of Figs. 4 and 5 must also be pure reactances. Under this assumption let us consider the free oscillations of Fig. 6; first, with K_2 assumed to be a pure reactance; second, with K_2 assumed to be a pure resistance; and third, in order to show that this third assumption is contrary to fact, with K_2 assumed to be an impedance with both resistance and reactance.

With K_2 a reactance, the circuit contains nothing but reactances, and free oscillations are possible if, and only if, the total impedance of the circuit is zero. The end impedances Z' and Z'' being different, the potentials at the ends of the mesh will be different, and this means that the corresponding wave on the infinite line will be attenuated, since the ratio between these potentials is the rate at which the amplitudes fall off per section.

With K_2 a pure resistance, a free oscillation is possible only if the dissipation in the positive resistance at the right end of the circuit is exactly made up by the hypothetical source of energy existing in the negative resistance $-K_2$ at the left end of the circuit. An exact balance between the energy supplied at one end and that lost at the other end is possible, since the equal positive and negative resistances K_2, $-K_2$ carry equal currents. This continuous transfer of energy from the left of the oscillating circuit of Fig. 6 to the right end is the action which goes on in every section of the infinite artificial line, and serves to pass forward the energy along the infinite line.

If K_2 were complex, $-K_2$ on the left of Fig. 6 and $+K_2$ on the right would not carry the same fraction of the circulating current I, since they are each shunted by a reactance $2Z_2$ which would allow less of the current to flow through $+K_2$ than through $-K_2$, if $2Z_2$ makes the smaller angle with $+K_2$, and vice versa. No balance between absorbed and dissipated energy is possible under these conditions when the equal and opposite resistance components carry unequal currents. A complex K_2, therefore, gives no free oscillation, and cannot occur with a resistanceless artificial line.

It is perhaps more instructive to consider the transmission on the line as a whole, rather than to confine attention exclusively to the oscillations of the simple circuit of Fig. 6 and so, at this point, without following further the conclusions to be drawn directly from this oscillating circuit, the fundamental energy theorem of resistanceless artificial lines will be stated, and then proved as a property of an infinite artificial line.

Energy Flow Theorem

Upon an infinite line of periodic recurrent structure, and devoid of resistance, a sinusoidal e.m.f. produces one of two steady states, viz.:

1. *A to-and-fro surging of energy without any resultant transfer of energy; currents and potential differences each attenuated from section to section, but everywhere in the same or opposite phase and mutually in quadrature, or,*

2. *A continuous, non-attenuated flow of energy along the line to infinity with no energy surging between symmetrical sections; current and potential non-attenuated, but retarded or advanced in phase from section to section, and mutually in phase at mid-shunt and mid-series points.*

The critical frequencies separating the two states of motion are the totality of the resonant frequencies of the series impedance, the anti-resonant frequencies of the shunt impedance, and the resonant frequencies of a single mid-shunt section of the line.

To prove the several statements of this theorem let us consider first the consequences of assuming that the wave motion, in progressing along the line, is attenuated, and next the consequences of assuming that the wave motion changes its phase. If the wave is attenuated, however little, at a sufficient distance it becomes negligible, and the more remote portions of the line may be completely removed without appreciable effect upon the disturbance in the nearer portion of the line. That part of the line which then remains is a finite network of pure reactances, and in any such network all currents are always in the same, or opposite, phase; so, also, are the potential differences; moreover, the two are mutually in quadrature; there is no continuous accumulation of energy anywhere, but only an exchange of energy back and forth between the inductances, the capacities and the generator. Continuously varying the amount of the assumed attenuation will cause a continuous variation in the corresponding frequency. The motion of the assumed character may, therefore, be expected to occur throughout continuous ranges or bands of frequencies and not merely at isolated frequencies.

The question may be asked—How far does the energy surge? Is the surge localized in the individual section, or does the surge carry the energy back and forth over more than one section, or even in and out of the line as a whole? To answer this question, it would be necessary, as we will now proceed to prove, to know something about the actual construction of the individual section. If each section is actually made up as shown in Fig. 6, and this is entirely possible in the present case (since only positive and negative reactances would be called for), then the section is capable of free oscillation, as explained above, and the surging is localized within the section; twice during each cycle the amount of energy increases on the right and decreases on the left. But we do not know that the section is made up like Fig. 6; we only know that it is equivalent to Fig. 6 as regards input and output relations. As far as these external relations go, the actual network may be made exclusively of either inductances or capacities with the connections shown in Fig. 4 or with the cross-connections of Fig. 5, according as the current is to have the same or opposite signs in consecutive sections. In any network made up exclusively of inductances or of capacities, the total energy falls to zero when the current or the potential falls to zero, respectively. Twice, therefore, in every cycle the total energy surges into this line and then it all returns to the generator. With other networks, surgings intermediate between these two extremes will occur. The theorem, therefore, does not limit the extent of the surging.

Under the second assumption, the phase difference between the currents at two given points, separated by a periodic interval, is to be an angle which is neither zero nor a multiple of $\pm\pi$. The assumed difference in phase can only be due to the infinite extension of the artificial line since, as previously noted, no finite sequence of inductances and capacities can produce any difference in phase. That infinite lines do produce phase differences is well-known; in particular, an infinite, uniform, perfectly conducting, metallic pair shows a continuous retardation in phase. If the infinitely remote sections of the artificial line are to have this controlling effect on the wave motion, the wave motion must actually extend to infinity, that is, there can be no attenuation. The wave progressing indefinitely to infinity without attenuation must be supplied continuously with energy; this energy must flow along the entire line with neither loss nor gain in the reactances it encounters on the way. This continuous flow of energy can take place only provided the currents and potentials are not in quadrature; they may be in phase. In considering the free oscillations of Fig. 6 it was shown that K_2 is real if it is not pure reactance. That is, for the mid-shunt section the current and potential are in phase. It is easy to show that they are also in phase at the mid-series point which is also a point of symmetry.

This flow-of-energy state of motion thus necessarily characterizes a phase-retarded wave on a resistanceless artificial line, regardless of the amount of the assumed positive or negative retardation, which may be taken to have any value between zero and exact opposition of phase. Continuously varying the retardation throughout the 180 degrees will, in general, call for a continuous change in the frequency of the wave motion. The second state of motion occurs, therefore, throughout continuous ranges or bands of frequencies.

No other state of motion is possible. With given initial amplitude and phase any possible wave motion is completely defined by its attenuation and phase change. All possible combinations of these two elements have been included in the two states, since the excluded conditions on each assumption have been included as a consequence of the other assumption. Thus, the exclusion of no attenuation in the first assumption was found necessarily to accompany the phase change of the second assumption; currents in phase or opposed, which were excluded from the second assumption, were found to be necessary features accompanying the first assumption. There remains only to consider the critical frequencies separating the two states of motion. At these frequencies there can be no attenuation and lag angles of multiples of $\pm\pi$, including zero, only. At symmetrical points the iterative impedance of the line must be a pure reactance to satisfy the first state of motion, and a pure resistance to satisfy the second state of motion. The only iterative impedances which satisfy these conditions are zero and infinity.

Some details relating to the pass and stop bands and the critical frequencies are brought together in the following table, where "stop ($\pm$)" refers to stop bands, the current being in phase or opposed in successive sections, and where γ and k refer to the line obtained by uniformly distributing $1/Z_2$ with respect to Z_1.

TABLE I.

For Ladder Artificial Line, Fig. 4

Band	Critical Frequency	Ratio $\frac{Z_1}{4Z_2}$	Uniform Line		Artificial Line			
			γ	k	Γ	$e^{-\Gamma}$	K_1	K_2
Stop (+)		>0	+real	imag.	+real	$0<<1$	imag.	imag.
	$Z_1=0$ $Z_2=\infty$	0 0	0 0	0 ∞	0 0	1 1	0 ∞	0 ∞
Pass		$0>>-1$	imag.	+real	imag.	$e^{i\theta}$	+real	+real
	$Z_1+4Z_2=0$	-1	$i2$	$i2Z_2$	$i\pi$	-1	0	∞
Stop (−)		<-1	imag.	+real	$i\pi$ + real	$-1<<0$	imag.	imag.
	$Z_1=\infty$ $Z_2=0$	$-\infty$ $-\infty$	∞ ∞	∞ 0	∞ ∞	0 0	∞ $\frac{1}{2}Z_1$	$2Z_2$ 0

It is not necessary to check the table item by item, many of which have already been proven, but it will be instructive to check some of the items by assuming that $Z_1/4Z_2$, called the ratio for brevity, is positive to begin with, and that a continuous increase in frequency reduces the ratio to zero and back through $\mp\infty$ to its original positive value. This cycle starts with a stop (+) band since the artificial line is in effect a network of reactances, all of which have the same sign; there is attenuation and the iterative impedances are imaginary. When the ratio decreases to zero, there must be either resonance which makes $Z_1 = 0$, or anti-resonance which makes $Z_2 = \infty$; in either case the artificial line has degenerated into a much simpler circuit; it is a shunt made up of all Z_2's combined in parallel, or a simple series circuit made up of all Z_1's, respectively; the iterative impedances are 0 and ∞, respectively; there is no attenuation in either case.

With a somewhat further increase of the frequency the ratio will assume a small negative value with the result that the artificial line will have both kinetic and potential energy. An analogy now exists between the artificial line and an ordinary uniform transmission line, which possesses both kinetic and potential energy, and is ordinarily visualized as being equivalent to many small positive reactances, in series, bridged, to the return conductor, by large negative reactances. The fact that uniform lines do freely transmit waves is a well-known physical principle, and it is not necessary to repeat here the physical theory of such transmission merely to show that the same phenomenon occurs with the identical structure when it is called an artificial line or wave-filter.

In order to determine just how far the ratio may depart from zero, on the negative side, without losing the property of free transmission, we look for any change in the action of the individual section of the artificial line which is fundamental; nothing less than a fundamental change in the behavior of the individual section can produce such a radical change in the line as an abrupt transition from the free transmission of a pass band to the to-and-fro surging of energy in a stop band. Now as the ratio is made more and more negative by the assumed increase of frequency, the value -1 is reached, at which frequency the symmetrical section (Fig. 6) of the artificial line is capable of free oscillation by itself. This is well recognized as a most fundamental change in the properties of any network, and it affords grounds for expecting a complete change in the character of the propagation over the artificial line. The change must be to a stop band with currents in opposite phase, since at resonance the potentials at the two ends of a section are in opposite phase.

Further increase in the frequency cannot make any change in the absolute difference in phase between the two ends of the other section, since opposition is the greatest possible difference in phase; the wave now adapts itself to increasing frequency by altering its attenuation.

Upon continuing the increase of frequency, so as to reduce the ratio to $-\infty$, we arrive at either anti-resonance corresponding to $Z_1 = \infty$ or resonance corresponding to $Z_2 = 0$; the artificial line has now degenerated into a row of isolated impedances Z_2, or into a series of impedances Z_1 short-circuited to the return wire; in either case the attenuation is infinite since no wave is transmitted. Passing beyond this critical frequency the ratio becomes positive, according to our assumption, and we are again in a stop (+) band.

While in this rapid survey of what happens during this frequency cycle little has been actually proven, it should have been made physically clear why abrupt changes in the character of the transmission occur at the frequencies making the ratio equal to 0, -1 or ∞, since the line degenerates into a simpler structure, or the phase change reaches its absolute maximum, on account of resonance, at these particular frequencies.

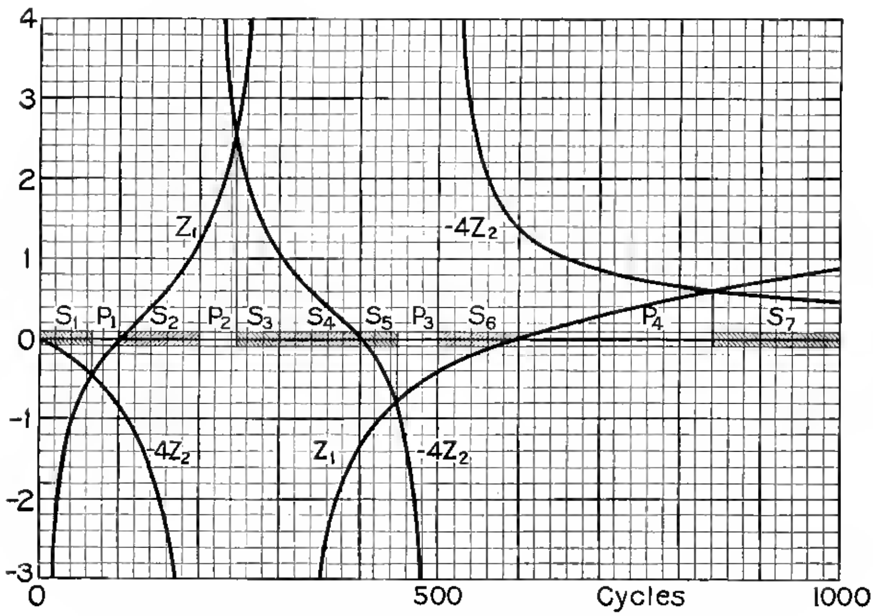


Fig. 7—Graph for Locating the Pass Bands and Stop Bands of Fig. 4
$Z_1 = -i(1^2 - x^2)(6^2 - x^2)/8x(3^2 - x^2)$,
$-4Z_2 = -i4x(4^2 - x^2)/(2^2 - x^2)(5^2 - x^2)$ and x = cycles/100

Information as to the location of the bands is often obtained most readily by plotting both Z_1 and $-4Z_2$, as illustrated in Fig. 7, and determining the critical frequencies by noting where the curves cross each other and the abscissa axis, as well as where they become infinite. Any particular band is then a pass band, a stop (+) band or a stop (−) band, according as Z_1, the abscissa axis, or $-4Z_2$ lies between the other two of the three lines. In Fig. 7 the pass bands are P_1, P_2, P_3, P_4; the stop (+) bands are S_2, S_4, S_6; and the stop (−) bands are S_1, S_3, S_5, S_7, and they illustrate quite a variety of sequences. By altering the curves the bands may be shifted, may be made to coalesce, or may be made to vanish.

Wave-Filter Curves

The pass band and stop band characteristics of wave-filters are concretely illustrated for a few typical cases by the curves of Figs. 8–13, which show the attenuation constant A, the phase constant B, and both the resistance R and reactance X components of the iterative impedance for a range of frequencies which include all of the critical frequencies, except infinity. The heavy curves apply to the ideal resistanceless case, while the dotted curves assume a power factor equal to $1/(20\pi)$ for each inductance which is a value readily obtained in practice. This value is, however, not sufficiently large to make these small scale curves entirely clear, since considerable portions of the dotted curves appear to be coincident with the heavy line curves; but this, as far as it goes, proves the value of the present discussion which rests upon a close approximation of actual wave-filters to the ideal resistanceless case.

The low pass resistanceless wave-filter, as shown by Fig. 8, presents no attenuation below 1,000 cycles; above this frequency the attenuation constant increases rapidly, in fact, the full line attenuation curve increases at the start with maximum rapidity, since it is there at right angles to the axis. The dotted attenuation curve, which includes the effective resistance in the inductance coils, follows the ideal attenuation curve closely, except in the neighborhood of 1,000 cycles, where resistance rounds off the abrupt corner which is present in the ideal A curve. The phase constant B is, at the start, proportional to the frequency, as for an ordinary uniform transmission line; its slope becomes steeper as the critical frequency 1,000 is approached where the curve reaches the ordinate π, at which value it remains constant for all higher frequencies. As shown by the dotted B curve, resistance rounds off the corner at the critical frequency, but

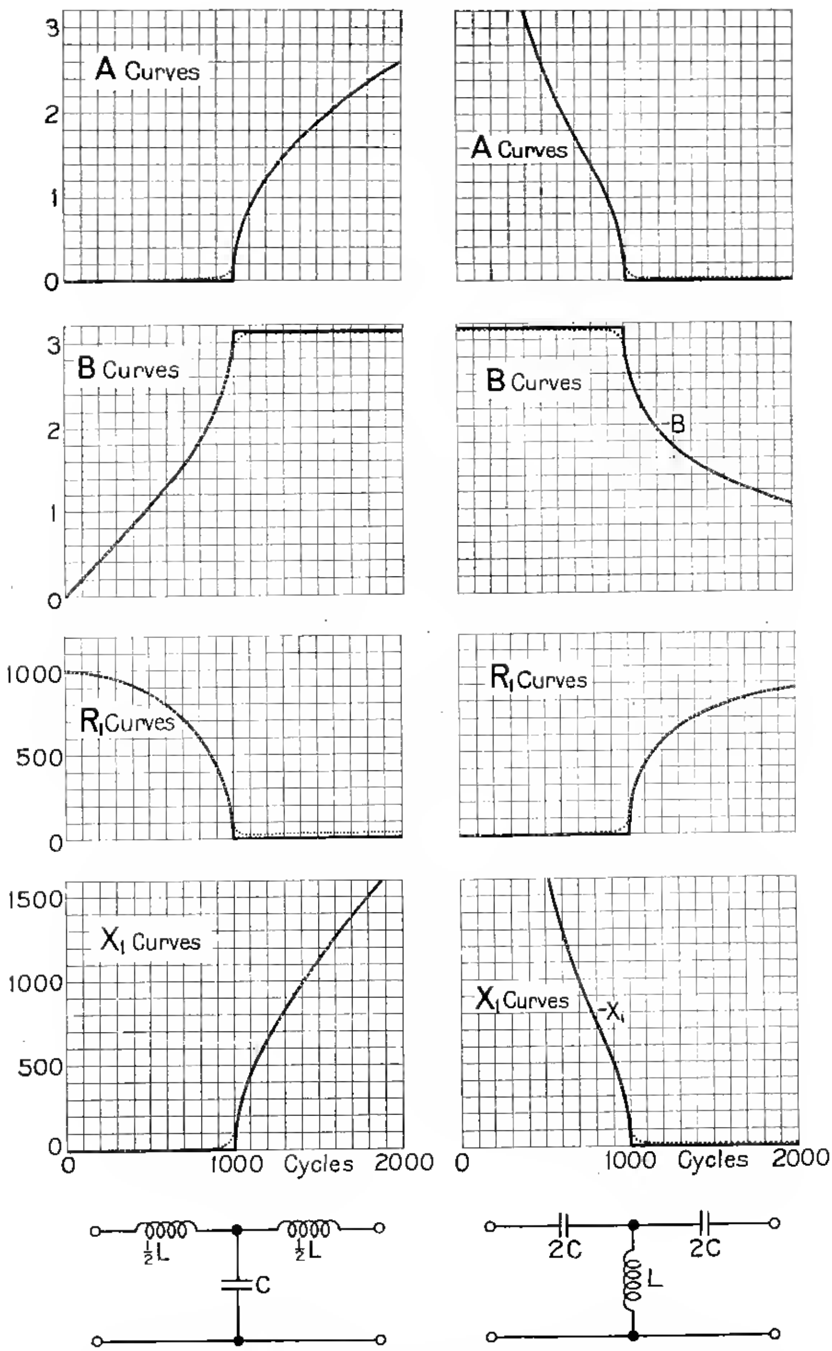


Fig. 8—Low Pass Wave-Filter: $L = 1/\pi$ Henry, $C = 1/\pi$ Microfarad
Fig. 9—Complementary High Pass Wave-Filter: $L = 1/4\pi$, $C = 1/4\pi$

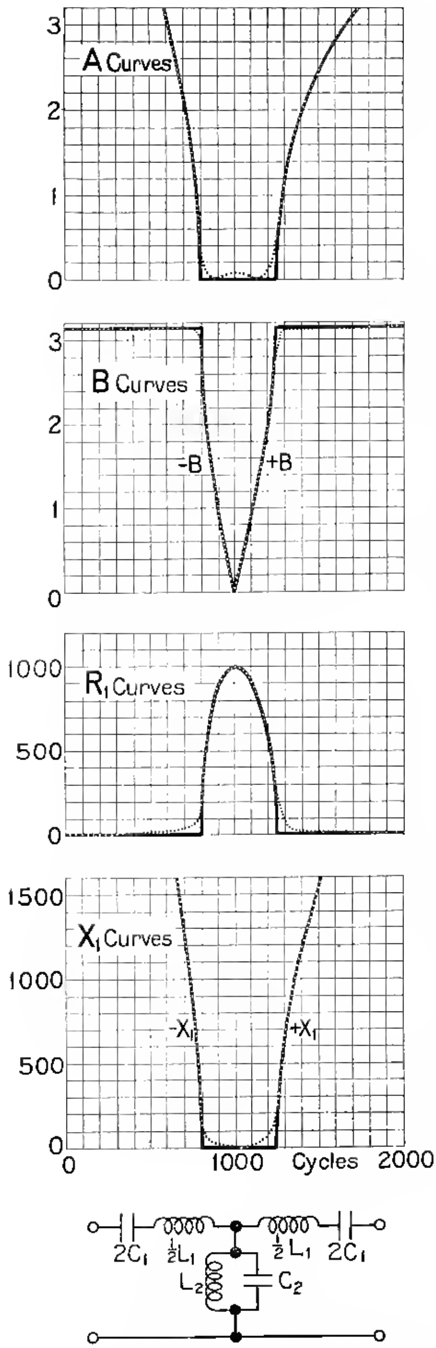
otherwise leaves the curve approximately unchanged. The full line curves for R_1 and X_1 show that in the ideal case the iterative impedance is pure resistance and pure reactance in the pass and stop bands respectively, and that resistance smooths the abrupt transition at the critical frequency.

The high pass wave-filter shown by Fig. 9 passes the band which is stopped by the low pass wave-filter of Fig. 8, and vice versa. For this reason the two wave-filters are said to be complementary.

Another set of two complementary wave-filters is shown by Figs. 10 and 11, one of which passes only a single band of frequencies, not extending to either zero or infinity, while the other passes the remaining frequencies only. The single pass band of Fig. 10, embracing a total phase change 2π on the B curve, is actually a case of confluent pass bands, each of which embraces the normal angle π. The tendency of the two simple pass bands to separate, and leave a stop band between them, is shown by the hump in the dotted attenuation constant curve at 1,000 cycles. If, instead of the two simple bands having been brought together, one of them had been relegated to zero or infinity, the single remaining pass band would have exhibited the normal angular range π in the B curve, and there would have been no hump in the dotted A curve. The stop band of Fig. 11 also illustrates peculiarities which are not necessary features of a wave-filter with a single stop band in this position. This wave-filter is obtained from Fig. 7 by making all bands vanish except P_2, S_3, S_5 and P_3,—by extending P_2 to zero, P_3 to infinity, and making S_3 and S_5 coalesce, so that the attenuation becomes infinite in the stop band without passing from a stop $(-)$ to a stop $(+)$ band. The coalescing stop bands are responsible for the rapid changes in the B, R_1, and X_1 curves of Fig. 11 which would not have appeared if, in Fig. 7, the same pass band had been obtained by retaining P_1, S_2 and P_2 and making all other bands vanish.

An extreme case of complementary wave-filters is shown by Figs. 12 and 13, where no frequencies and all frequencies are passed respectively. The first result is obtained by combining inductances alone, which, as has been pointed out above, can give only an attenuated disturbance devoid of wave characteristics. The wave-filter shown for passing all frequencies has inductance coils in the line, and capacities diagonally bridged across the line. This wave-filter combines a constant iterative impedance with a progressive change in phase which is sometimes useful.[4] An outstanding char-

[4] A theoretical use of the phase shifting afforded by the lattice artificial line was made at page 253 of "Maximum Output Networks for Telephone Substation and Repeater Circuits," Trans. A. I. E. E., vol. 39, pp. 231–280, 1920.



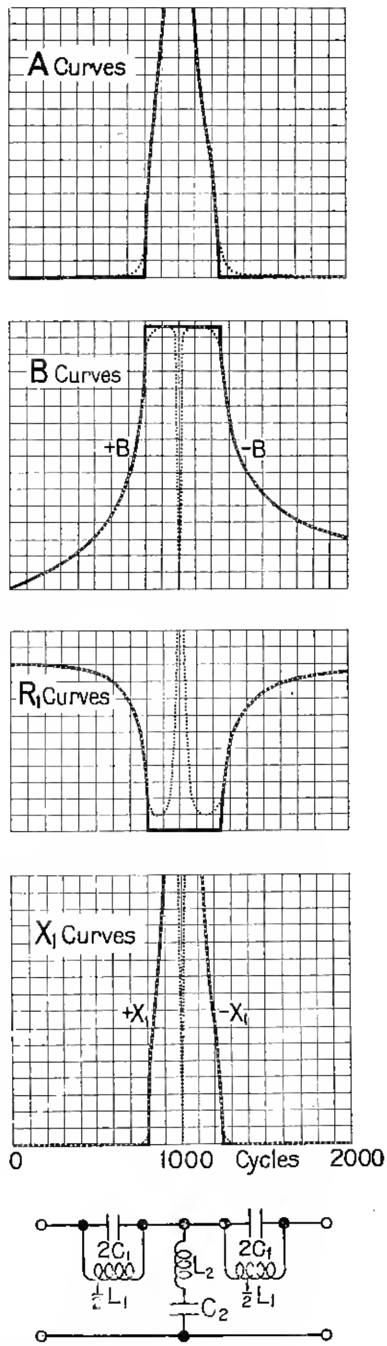
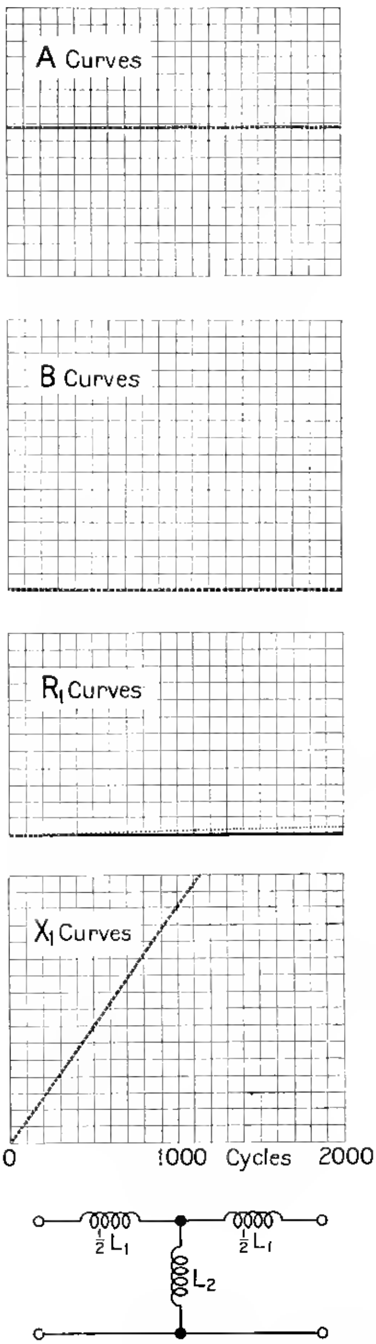


Fig. 10—Single Band Pass Wave-Filter: $L_1 = 20/9\pi$, $L_2 = 9/80\pi$, $C_1 = 9/80\pi$, $C_2 = 20/9\pi$

Fig. 11—Complementary High and Low Pass Wave-Filter: $L_1 = 9/20\pi$, $L_2 = 5/9\pi$, $C_1 = 5/9\pi$, $C_2 = 9/20\pi$



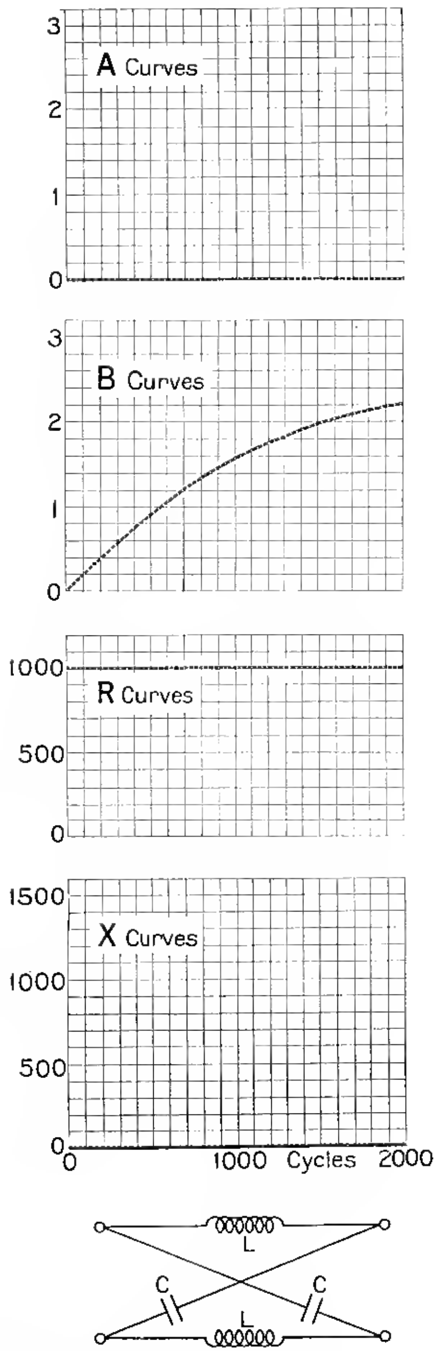


Fig. 12—No Pass Wave-Filter: $L_1 = 1/\pi$, $L_2 = 1/4\pi$
Fig. 13—Complementary All Pass Wave-Filter: $L = 1/2\pi$, $C = 1/2\pi$

acteristic of this type of artificial line is that it has, for all frequencies, the same iterative impedance as a uniform line with the same total series and shunt impedances. This artificial line will be considered in more detail in the next section of this paper.

Lattice Artificial Lines

Up to this point we have considered the properties of artificial line networks which were supposed to be given. In practice the problem is ordinarily reversed, and we ask the questions: May the locations of the bands be arbitrarily assigned? May additional conditions be imposed? How may the corresponding network be determined, and what is its attenuation in terms of the assigned critical frequencies? These questions might be answered by a study of Fig. 7, in

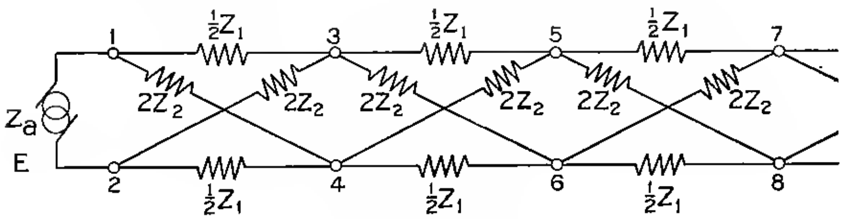


Fig. 14—Lattice Artificial Line

all its generality, but it seems simpler to base the discussion upon the artificial line shown in Fig. 14, which is to be a generalization of Fig. 13 to the extent of making the two impedances Z_1 and Z_2 any possible actual driving-point impedances. It is sometimes illuminating to regard this artificial line as a nest of bridges, one within another, as shown by Fig. 15.

On interchanging terminals 3 with 4 and 7 with 8 in Fig. 14 the network of lines remains unchanged; thus, Z_1 and $4Z_2$ may be interchanged in the formulas for the artificial line with no change in the result, except, possibly, one corresponding to a reversal of the current at alternate junction points. Another elementary feature of this artificial line is that it degenerates into a simple shunt or a simple series circuit at the resonant or anti-resonant frequencies, respectively, of either Z_1 or Z_2, and these are the critical frequencies, terminating the pass bands. At other frequencies, a positive ratio $Z_1/4Z_2$ must give a stop band, since the reactances are all of one sign. If a small negative value of this ratio gives free transmission, as we naturally expect, there will be identical transmission, except for a reversal of

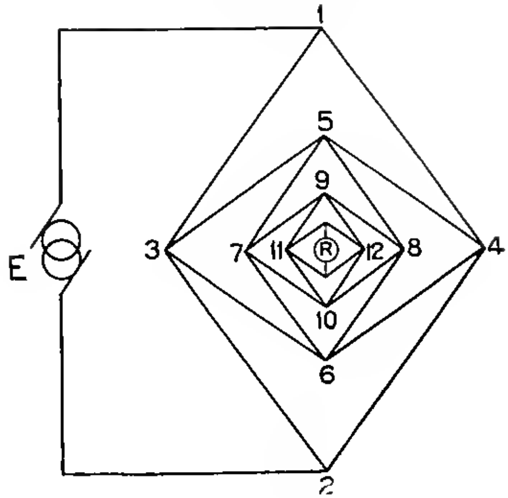


Fig. 15—Lattice Artificial Line Drawn to Show the Chain of Bridge Circuits

sign, when the ratio has the reciprocal value, which will be a large negative quantity, since we may always interchange Z_1 and $4Z_2$. The consequences of this and of other elementary properties of this artificial line are brought together in the following table:

TABLE II

For Lattice Artificial Line, Fig. 14

Band	Critical Frequency	Ratio $\frac{Z_1}{4Z_2}$	UNIFORM LINE		ARTIFICIAL LINE		
			γ	k	Γ	$e^{-\Gamma}$	K
Stop (+)		$1>>0$	$2>>0$	imag.	+ real	$0<<1$	imag.
	$Z_1 = 0$ $Z_2 = \infty$	0 0	0 0	0 ∞	0 0	1 1	0 ∞
Pass		<0	imag.	+ real	imag.	$e^{i\theta}$	+ real
	$Z_1 = \infty$ $Z_2 = 0$	∞ ∞	∞ ∞	∞ 0	$i\pi$ $i\pi$	-1 -1	∞ 0
Stop (−)		<1	<2	imag.	$i\pi$ + real	$-1<<0$	imag.
	$Z_1 = 4Z_2$	1	2	$2Z_2$	∞	0	$2Z_2$

The cycle of bands: stop (+), pass, stop (−), adopted for the table, carries the attenuation factor $e^{-\Gamma}$ around the periphery of a unit semi-circle; in the stop (+) band it traverses the radius from 0 to 1, in the pass band it travels along the unit circle through 180 degrees to the value -1, completing the cycle from -1 to 0 in the stop ($-$) band. In this cycle there are four points of special interest, corresponding to ratio values 1, 0, -1 and ∞, for which the wave is infinitely attenuated, unattenuated with an angular change of 0, of 90, and of 180 degrees, respectively. It is at the 90 degree angle that resonance of the individual section occurs; the iterative impedance is then equal to $2|Z_2|$.

Graph of the Ratio $Z_1/4Z_2$ for Fig. 14

If we plot Z_1 and $4Z_2$ the pass bands are shown by the points where the curves become zero or infinite, and the intersections of the two

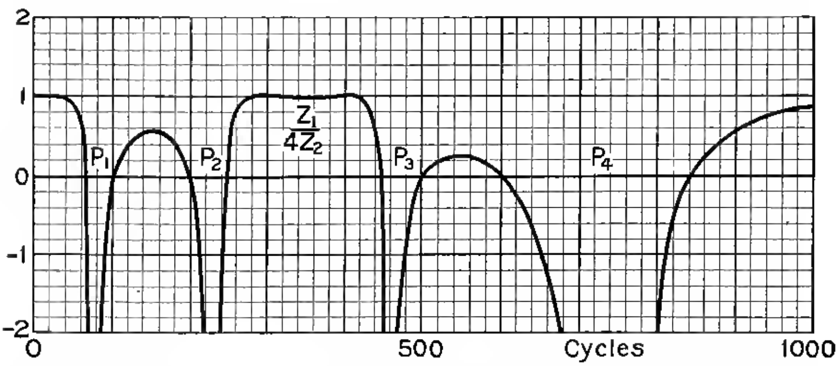


Fig. 16—Graph for Locating the Pass and Stop Bands of the Lattice Artificial Line, where $Z_1/4Z_2 = \left[(x_1^2 - x^2)\,(x_2^2 - x^2)^{-2}\,(x_3^2 - x^2)\right]\ldots$, x = cycles/100, and the resonant roots $x_1, x_3, \ldots$ are 0.650, 1, 2, 2.452, 4.442, 5, 6, 8.476 and the double anti-resonant roots $x_2, x_5 \ldots$ are 0.766, 2.301, 4.585, 7.423

curves show the frequencies at which the attenuation becomes infinite. These intersections must be at an acute angle since each branch of the two curves has a positive slope throughout its entire length; for this reason it may be desirable to plot the ratio rather than the individual curves; this is especially desirable in cases where the two curves do not intersect, but are tangent. Fig. 16 is for a lattice network equivalent to two sections of the ladder type illustrated by Fig. 7, and so cannot include a stop ($-$) band. Accordingly, the ratio does not go above unity, although it reaches unity at the two frequencies 300 and 400, corresponding to the infinite attenuation where stop ($-$) and stop (+) bands meet in Fig. 7. It is also unity at the extreme frequencies zero and infinity. The four pass bands have, of course, the same locations as in Fig. 7.

Multiplying the ratio by a constant greater than unity introduces stop (−) bands along with the stop (+) bands; multiplying it by a constant less than unity removes all infinite attenuations; these changes within the stop bands are made without altering the locations of the four pass bands.

Wave-Filter Having Assigned Pass Bands

In connection with practical applications we especially desire to know what latitude is permitted in the preassignment of properties for a wave-filter. If we consider first the ideal lattice wave-filter, its limitations are those inherent in the form which its two independent resistanceless one-point impedances[5] Z_1 and Z_2 may assume. The mathematical form of this impedance is shown by formula (7) of the appendix, which may be expressed in words as follows:

Within a constant factor the most general one-point reactance obtainable by means of a finite, pure reactance network is an odd rational function of the frequency which is completely determined by assigning the resonant and anti-resonant frequencies, subject to the condition that they alternate and include both zero and infinity.

The corresponding general expressions for the quotient and product of the impedances Z_1 and Z_2 are shown by formulas (8) and (9). Definite, realizable values for all of the $2n+2$ parameters and $2n+1$ optional signs occurring in these formulas may be determined in the following manner:

(a) Assign the location of all n pass bands, which must be treated as distinct bands even though two or more are confluent; this fixes the values of the $2n$ roots $p_1 \ldots p_{2n}$ which correspond to the successive frequencies at the two ends of the bands.

(b) Assign to the lower or upper end of each pass band propagation without phase change from section to section; this fixes the corresponding optional sign in formula (8) as + or −, respectively.

(c) Assign a value to the propagation constant at any one non-critical frequency (that is, assign the attenuation constant in a

[5] A one-point impedance of a network is the ratio of an impressed electromotive force at a point to the resulting current at the same point—in contradistinction to two-point impedances, where the ratio applies to an electromotive force and the resulting current at two different points.

stop band or the phase constant in a pass band); this fixes the value of the constant G and thus completely determines formula (8) on which the propagation constant depends.

(d) Assign to the lower or upper end of each stop band the iterative impedance zero; this fixes the corresponding optional sign in formula (9) as $+$ or $-$, respectively.

(e) Assign the iterative impedance at any one non-critical frequency (subject to the condition that it must be a positive resistance in a pass band and a reactance in a stop band); this fixes the constant H and thereby the entire expression (9) upon which the iterative impedance depends.

The quotient and product of the impedances Z_1 and Z_2 are now fully determined; the values of Z_1 and Z_2 are easily deduced and also the propagation constant and iterative impedance by formulas (11) and (12); Z_1 and Z_2 are physically realizable except for the necessary resistance in all networks.

These important results may be summarized as follows:

A lattice wave-filter having any assigned pass bands is physically realizable; the location of the pass bands fully determines the propagation constant and iterative impedance at all frequencies when their values are assigned at one non-critical frequency, and zero phase constant and zero iterative impedance are assigned to the lower or upper end of each pass band and stop band, respectively.

Lattice Artificial Line Equivalent to the Generalized Artificial Line of Fig. 1

Since any number of arbitrarily preassigned pass bands may be realized by means of the lattice network, it is natural to inquire whether this network does not present a generality which is essentially as comprehensive as that obtainable by means of any network N in Fig. 1, provided the generalized line is so terminated as to equalize its iterative impedances in the two directions. This proves to be the case.

If network N has identical iterative impedances in both directions, the lattice network equivalent to two sections of N is shown by Fig. 17; each lattice impedance is secured by using an N network; the N's placed in the two series branches of the lattice have their far terminals short-circuited so that they each give the impedance denoted by Z_0; the N's in the two diagonal branches have their far ends open and they each give the impedance denoted by Z_∞.

The lattice network of Fig. 18 has in each branch a one-point impedance obtained by means of a duplicate of the given network N and an ideal transformer. The two lattice branch impedances are $Z_q+Z_r\pm 2Z_{qr}$ where the three impedances Z_q, Z_r, Z_{qr} are the effective self and mutual impedances of the network N regarded as a transformer. This lattice network has identically the same propaga-

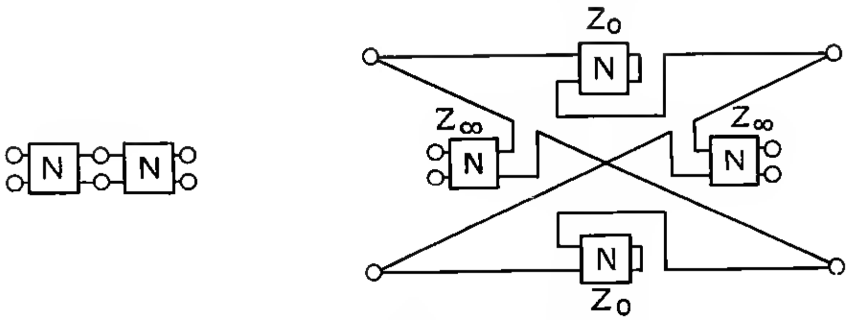


Fig. 17—Lattice Unit Equivalent to Two Sections of Fig. 1 Assumed to be Symmetrical

tion constant as the single network N shown on the left. Since the lattice cannot have different iterative impedances in the two directions, it actually compromises by assuming the sum of the two iterative impedances presented by N. A physical theory of the equival-

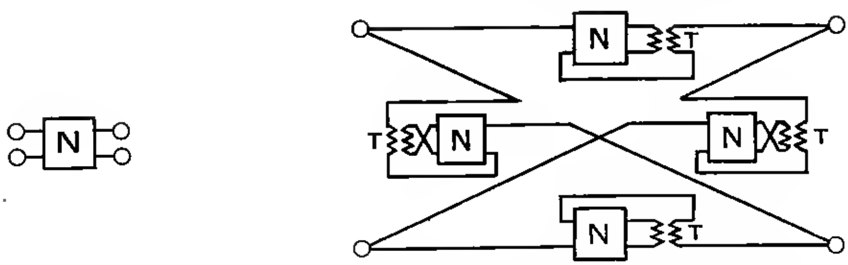


Fig. 18—Lattice Network Having the Same Propagation Constant as N and an Iterative Impedance Equal to the Sum of the Two Iterative Impedances of N

ences shown in Figs. 17 and 18 has not been worked up; the analytical proofs were made by applying the formulas given in the appendix under lattice networks.

Without going to more complex networks it is, of course, not possible to get a symmetrical iterative impedance, but that is not necessary for our present purposes where we are concerned primarily with the propagation constant. It has now been shown with complete generality that:

The lattice artificial line, with physically realizable branch impedances, is identically equivalent in propagation constant and mean iterative impedance to the chain of identical physically realizable networks connected together in sequence through two pairs of terminals.

To complete this simplification of the generalized artificial line it is necessary to know the simplest possible form of the one-point impedances employed in the branches of the lattice network. The discussion of the most general one-point impedance obtainable by means of any network of resistances, self and mutual inductances, leakages and capacities will find its natural place, together with allied theorems, in a paper on the subject of impedances. For the present purpose it is sufficient to state:

The most general branch impedance of the lattice network may be constructed by combining, in parallel, resonant circuits having impedances of the form $R+iLp+(G+iCp)^{-1}$; or they may equally well be constructed by combining, in series, anti-resonant circuits having impedances of the form $\left[G+iCp+(R+iLp)^{-1}\right]^{-1}$.

Summary of Physical Theory

The wave-filter under discussion approximates to a resistanceless artificial line, and such an ideal artificial line is capable of two, and only two, fundamentally distinct states of motion. In one state the disturbance is attenuated along the line, and there is no flow of energy other than a back and forth surging of energy, the intensity of which rapidly dies out along the line. In the other state there is a free flow of energy, without loss, from section to section along the line, with no surge of energy between symmetrical sections. Each state holds for one or more continuous bands of frequencies; these bands have been distinguished as stop bands and pass bands.

A high degree of discrimination, between different frequencies, may be obtained, even if each section, taken alone, gives only a moderate difference in attenuation, by the use of a sufficient number of sections in the wave-filter, since the attenuation factors vary in geometrical progression with the number of sections.

Any number of arbitrarily located pass bands may be realized by means of the lattice artificial line; furthermore, the propagation constant at one frequency, and the iterative impedance at one frequency may both be assigned, while the location of zero phase constant and zero iterative impedance at the lower or upper end of each pass band and stop band, respectively, is also optional. This completely determines the lattice artificial line. No additional condition, other than iterative impedance asymmetry, can be realized by replacing the lattice network by any four terminal network.

APPENDIX

Formulas for the Artificial Line

Formulas for the propagation constant and iterative impedance of the generalized artificial line, expressed in a number of equivalent forms, have already been given in my paper on Cisoidal Oscillations,[6] but it seems worth while to deduce the formulas anew here from the free oscillations of the detached unit circuit of Fig. 6, so as to complete the physical theory by deducing the comprehensive mathematical formulas by the same method of procedure.

Ladder Network Formulas

Notation:

Z_1, Z_2 = series impedance and shunt impedance of the section of Fig. 4, which is equivalent to the general network N of Fig. 1.

$\Gamma = A + iB$ = propagation constant per section.

K_1, K_2 = iterative impedances at mid-series and mid-shunt.

$\gamma = \alpha + i\beta = \sqrt{Z_1/Z_2}$ = propagation constant for uniform distribution of Z_1 and $1/Z_2$, per unit length.

$k = \sqrt{Z_1 Z_2}$ = iterative impedance of this same uniform line.

In Fig. 6, the current is indicated as I and the potentials at the ends of the section as $V, Ve^{-\Gamma}$. In order that the free oscillation may be possible the total impedance of the circuit $(Z_1 + Z' + Z'')$ must vanish; this determines the iterative impedance K_2. In addition to this condition it is sufficient to make use of two other simple relations: the proportionality of the potential drops in the direction of the current across Z' and Z'' to Z' and Z'', since they carry the same current (this determines the propagation constant Γ); and the equality of

[6] "Cisodial Oscillations," Trans. A. I. E. E., vol. 30, pp. 873–909, 1911. In the lowest row of squares of Table I, the iterative impedances and propagation constant of any network are given in five different ways, involving one-point and two-point impedances, equivalent star impedances, equivalent delta impedances, equivalent transformer impedances, or the determinant of the network. The only typographical errors in Table I appear to be the four which occur in the first, third and fifth squares of this row: in the values for K_q replace $(S_q - S_{qr})$ by $(S_q - S_r)$ and place a parenthesis before $U_q - U_r)$; in the first value of K_r replace S_{qr} by S_{qr}^2; in the last value for Γ_{qr} add a minus sign so that it reads $\cosh^{-1}$.

K_1, the mid-series iterative impedance of the artificial line, to the total impedance on the right of the mid-point of the series impedance Z_1. These three relations, which can be written down at once, are:

$$Z_1 + Z' + Z'' = Z_1 - \frac{4Z_2K_2^2}{4Z_2^2 - K_2^2} = 0,$$

$$\frac{Ve^{-\Gamma}}{V} = -\frac{Z''}{Z'} = \frac{2Z_2 - K_2}{2Z_2 + K_2},$$

$$K_1 = \frac{1}{2}Z_1 + Z'' = \frac{1}{2}Z_1 + \frac{2Z_2K_2}{2Z_2 + K_2},$$

from which the formulas for Γ, K_1, and K_2, in terms of Z_1, Z_2, are found to be:

$$\Gamma = 2\sinh^{-1}\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2\sinh^{-1}\frac{1}{2}\gamma, \quad (1)$$

$$\left.\begin{matrix} K_1 \\ K_2 \end{matrix}\right\} = \sqrt{Z_1Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)^{\pm\frac{1}{2}} = k\left(1 + \frac{1}{4}\gamma^2\right)^{\pm\frac{1}{2}} \text{ at mid } \left\{\begin{matrix} \text{series} \\ \text{shunt,} \end{matrix}\right. \quad (2)$$

and the formulas for Z_1 and Z_2 in terms of Γ and K_1 or K_2 are likewise found to be:

$$Z_1 = 2K_1 \tanh\frac{1}{2}\Gamma = K_2 \sinh \Gamma, \quad (3)$$

$$Z_2 = K_1/\sinh \Gamma = \frac{1}{2}K_2 \coth\frac{1}{2}\Gamma. \quad (4)$$

Formulas (3) and (4) are in the nature of design formulas in that they determine the impedance Z_1 and Z_2, at assigned frequencies, which will ensure the assigned values of Γ and K at these frequencies. In general, however, it would not be evident how best to secure these required values of Z_1 and Z_2; complicated or even impossible networks might be called for, even to approximate values of Z_1 and Z_2 assigned in an arbitrary manner. Fortunately, practical requirements are ordinarily satisfied by meeting maximum and minimum values for the attenuation constant throughout assigned frequency bands. Formulas (8) and (9) may be employed for this purpose as explained below.

It is convenient to have formulas (1) and (2) expressed in a variety of ways, since no one form is well suited for calculation throughout the entire range of the variables. Accordingly, the following analytically equivalent expressions are here collected together for reference:

$$\Gamma = i\,2\sin^{-1}\frac{\gamma}{2i} = i\cos^{-1}\left(1+\frac{\gamma^2}{2}\right), \tag{5}$$

$$= 2\sinh^{-1}\frac{\gamma}{2} = 2\tanh^{-1}\frac{\frac{\gamma}{2}}{\sqrt{1+\frac{\gamma^2}{4}}} = \cosh^{-1}\left(1+\frac{\gamma^2}{2}\right)$$

$$= 2\log\left[\frac{\gamma}{2}+\sqrt{1+\frac{\gamma^2}{4}}\right], \tag{5a}$$

$$= i\pi + 2\cosh^{-1}\frac{\gamma}{2i} = i\pi + \cosh^{-1}\left(-1-\frac{\gamma^2}{2}\right)$$

$$= i\pi + 2\log\left[\frac{\gamma}{2i}+\sqrt{-1-\frac{\gamma^2}{4}}\right], \tag{5b}$$

$$= i\frac{\pi}{2} - \sinh^{-1}\left(1+\frac{\gamma^2}{2}\right)i, \tag{5c}$$

$$= \gamma - \frac{1}{24}\gamma^3 + \frac{3}{640}\gamma^5 - \frac{5}{7168}\gamma^7 + \ldots, \text{ if } |\gamma| < 2, \tag{5d}$$

$$= 2\cosh^{-1} g + i\,2\sin^{-1}\frac{\beta}{2g}, \tag{5e}$$

where $\gamma = \alpha + i\beta,\ 2g = \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(1+\frac{\beta}{2}\right)^2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(1-\frac{\beta}{2}\right)^2}$,

$$= \cosh^{-1} h + i\cos^{-1}\frac{x}{h}, \tag{5f}$$

where $1 + \frac{1}{2}\gamma^2 = x + iy,\ 2h = \sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2}$,

$$\left.\begin{matrix}K_1\\K_2\end{matrix}\right\} = k\left(1+\frac{1}{4}\gamma^2\right)^{\pm\frac{1}{2}} = k\left(\cosh\frac{\Gamma}{2}\right)^{\pm 1} = k\left(\frac{\sinh\Gamma}{\gamma}\right)^{\pm 1}$$

$$= k\left(\frac{1}{2}\gamma\coth\frac{1}{2}\Gamma\right)^{\pm 1} \text{ at mid } \left\{\begin{matrix}\text{series}\\\text{shunt.}\end{matrix}\right. \tag{6}$$

The formulas leave indeterminate the signs of γ, k, Γ, and K, and also a term $\pm i2\pi n$ in γ and Γ. The signs are to be so chosen that the real parts are positive, or become positive when positive resistance is added to the system. The indeterminate $\pm\, i2\pi n$ can be made determinate only after knowing something of the internal structure of the unit network of which the artificial line is composed; the conditions to be met are—absence of phase differences when all branches of the unit network N of Fig. 1 are assumed to be pure resistances and continuity of phase as reactances are gradually introduced to give the actual network.

Formula (5) is adapted for use in the pass bands, since the expressions are real when γ^2 is real, negative and not less than -4; similarly, formulas (5a) and (5b) are adapted for use in the stop $(\pm)$ bands, that is, when γ^2 is positive and less than -4 respectively.

From the theory of impedances we know that any resistanceless one-point impedance is expressible in the form

$$Z = iD \frac{p \quad (p_2^2-p^2) \ldots (p_{2n-2}^2-p^2)}{(p_1^2-p^2)\ (p_3^2-p^2) \ldots (p_{2n-1}^2-p^2)} \tag{7}$$

where the factor D and the roots $p_1, p_2, \ldots p_{2n}$ are arbitrary positive, reals subject only to the condition that each root is at least as large as the preceding one. This enables us to write down the forms which the quotient and product of two resistanceless one-point impedances may assume, which are as follows:

$$\frac{Z'}{Z''} = G\left(\frac{p_1^2-p^2}{p_2^2-p^2}\right)^{\pm 1}\left(\frac{p_3^2-p^2}{p_4^2-p^2}\right)^{\pm 1} \ldots \left(\frac{p_{2n-1}^2-p^2}{p_{2n}^2-p^2}\right)^{\pm 1} \tag{8}$$

$$Z'Z'' = -H\left(\frac{p^2}{p_1^2-p^2}\right)^{\pm 1}\left(\frac{p_2^2-p^2}{p_3^2-p^2}\right)^{\pm 1} \ldots \left(\frac{p_{2n-2}^2-p^2}{p_{2n-1}^2-p^2}\right)^{\pm 1}(p_{2n}^2-p^2)^{\pm 1} \tag{9}$$

where G, H and the roots $p_1, p_2, \ldots p_{2n}$ are arbitrary positive reals, subject only to the condition that each root is at least as large as the preceding one, and the $2n+1$ and optional $\pm$ signs are mutually independent. Conversely, if the relations (8) and (9) are prescribed, then the required individual impedances Z' and Z'' are each of the form (7) and thus physically realizable.

If in formulas 1, 2, 5 and 6 we substitute for $Z_1/Z_2 = \gamma^2$ and $Z_1 Z_2 = k^2$ the right-hand side of formulas (8) and (9), respectively, we obtain formulas for the propagation constant and iterative impedance of an artificial resistanceless line in terms of frequencies at which the propagation constant becomes zero or infinite. Ordinarily, however, we are more interested in having expressions in terms of the frequencies which terminate the pass bands. To secure these the substitutions should be $4[8]/(4-[8])$ and $[9]\ (1-[8]/4)^{\pm 1}$, where [8] and [9] stand for the entire right-hand sides of formulas (8) and (9). This substitution amounts to obtaining the lattice network giving the required pass bands, and then transforming to the ladder network having the same propagation constant and the same iterative impedance at mid-series or mid-shunt.

Lattice Network Formulas Fig. 14

The impedances of a single section between terminals 1 and 2, with the far end of the section 3 and 4 either short-circuited or open, are readily seen to be

$$Z_0 = \frac{2Z_1Z_2}{\frac{1}{2}Z_1 + 2Z_2}, \quad Z_\infty = \frac{1}{2}\left(\frac{1}{2}Z_1 + 2Z_2\right). \tag{10}$$

Since $\sqrt{Z_0Z_\infty}$ and $\sqrt{Z_0/Z_\infty}$ are the iterative impedance and the hyperbolic tangent of the propagation constant for any symmetrical artificial line, we have the following analytically equivalent formulas for the lattice network where $\gamma = \sqrt{Z_1/Z_2}$, and $k = \sqrt{Z_1Z_2}$ as for the ladder type.

Lattice Formulas

$$\Gamma = 2\tanh^{-1}\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2\tanh^{-1}\frac{1}{2}\gamma, \tag{11}$$

$$K = \sqrt{Z_1Z_2} = k. \tag{12}$$

$$Z_1 = 2K\tanh\frac{1}{2}\Gamma, \tag{13}$$

$$Z_2 = \frac{1}{2}K\coth\frac{1}{2}\Gamma. \tag{14}$$

$$\Gamma = i2\tan^{-1}\frac{\gamma}{2i} = i\cos^{-1}\frac{1+\frac{1}{4}\gamma^2}{1-\frac{1}{4}\gamma^2}, \tag{15}$$

$$= 2\tanh^{-1}\frac{\gamma}{2} = 2\sinh^{-1}\frac{\frac{1}{2}\gamma}{\sqrt{1-\frac{1}{4}\gamma^2}} = \cosh^{-1}\frac{1+\frac{1}{4}\gamma^2}{1-\frac{1}{4}\gamma^2}$$

$$= \log\frac{1+\frac{1}{2}\gamma}{1-\frac{1}{2}\gamma}, \tag{15a}$$

$$= i\pi + 2\coth^{-1}\frac{\gamma}{2} = i\pi + \cosh^{-1}\frac{1+\frac{1}{4}\gamma^2}{-1+\frac{1}{4}\gamma^2}$$

$$= i\pi + \log\frac{1+\frac{1}{2}\gamma}{-1+\frac{1}{2}\gamma}, \qquad (15b)$$

$$= i\frac{\pi}{2} - \sinh^{-1}\frac{1+\frac{1}{4}\gamma^2}{1-\frac{1}{4}\gamma^2}\,i, \qquad (15c)$$

$$= \gamma + \frac{1}{12}\gamma^3 + \frac{1}{80}\gamma^5 + \frac{1}{448}\gamma^7 + \ldots, |\gamma| < 2, \qquad (15d)$$

$$= \frac{1}{2}\log\frac{\left(1+\frac{\alpha}{2}\right)^2 + \left(\frac{\beta}{2}\right)^2}{\left(1-\frac{\alpha}{2}\right)^2 + \left(\frac{\beta}{2}\right)^2} + i\tan^{-1}\frac{\beta}{1-\left(\frac{\alpha}{2}\right)^2 - \left(\frac{\beta}{2}\right)^2}, \qquad (15e)$$

where $\gamma = \alpha + i\beta$.

In these formulas $Z_1/Z_2 = \gamma^2$ and $Z_1 Z_2 = k^2$ might be expressed in terms of the resonant and anti-resonant complex frequencies of Z_1 and Z_2, the frequencies being made complex quantities so as to include the damping. Where there is no damping, that is, where all network impedances are devoid of resistance, the simplified forms of these expressions are given by formulas (8) and (9). The use of these formulas for designing wave filters having assigned pass bands is explained at page 23.